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Synthetic Hamiltonian Mechanics

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This paper deals with some infinitesimal aspects of Hamiltonian mechanics from the standpoint of synthetic differential geometry. Fundamental results concerning Hamiltonian vector fields, Poisson brackets, and momentum mappings are discussed. The significance of the Lie derivative in the synthetic context is also consistently stressed. In particular, the notion of an infinitesimally Euclidean space is introduced, and the Jacobi identity of vector fields with respect to Lie brackets is established naturally for microlinear, infinitesimally Euclidean spaces by using Lie derivatives instead of a highly combinatorial device such as P. Hall's 42-letter identity.

INTRODUCTION

We know well that infinitesimals were vivid and active in the realm of analysis during the days of Newton and Leibniz and that they were rampant throughout the works of such pioneers in differential geometry as Cartan, Lie, and Riemann. The so-called $\epsilon - \delta$ argument has made mathematics rigorous by eradicating infinitesimals relentlessly, so that the majority of contemporary mathematicians prefer to turn them down as anathema or, at best, to leave them in oblivion.

Nowadays infinitesimals are coming back from mythology through two streams. One is nonstandard analysis, which is an application of model theory to analysis. For nonstandard analysis, the reader is referred, e.g., to Robinson (1966) and Stroyan and Luxemburg (1976). The other is synthetic differential geometry as championed by Lawvere, Kock, and others. Distinct from the former, it deals not only with invertible infinitesimals, but also with nilpotent ones. It gives a solid foundation to such once-dubious expressions as "vector fields are infinitesimal transformations." For synthetic differential geometry,

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the reader is referred, e.g., to Kock (1981), Lavendhomme (1987), and Moerdijk and Reyes (1991).

As is well known, Newtonian mechanics has been transmogrified, by such great pioneers as Lagrange, Laplace, Hamilton, Jacobi, Poisson, etc., into analytical mechanics, which consists of two principal branches, namely Lagrangian and Hamiltonian mechanics. The former is based on variational principles and can be generalized readily into a general relativistic context. The latter is based on the energy concept and has a great deal to do with quantum mechanics. For classical texts on analytical mechanics, the reader is referred, e.g., to Goldstein (1980) and Whittaker (1961).

Recently Hamiltonian mechanics has obtained a mathematically sophisticated arena, namely, symplectic manifolds, so that it could purport to be a branch of differential geometry. For Hamiltonian mechanics on symplectic manifolds, the reader is referred, e.g., to Abraham and Marsden (1978), Marsden and Ratiu (1994), and Puta (1993). The principal objective of this paper is to develop its rudiments from the standpoint of synthetic differential geometry. Synthetic Lagrangian mechanics will be discussed in a subsequent paper.

The organization of the paper goes as follows: The remainder of the paper is divided into two sections. Consisting of six subsections, the first section is concerned with synthetic differential geometry and forms prerequisites for the second section, concerned with synthetic Hamiltonian mechanics. The leading two subsections of the first section are completely a review on the set R of real numbers and microlinear spaces as its generalization. The succeeding two subsections deal with vector fields and differential forms, both of which have been investigated at large in the literature. However, our coherent emphasis on the usefulness of Lie derivative in synthetic context seems fresh and worthwhile. In Section 1.3, to make the calculus of Lie derivatives legitimate, we should assume that the totality of vector fields forms a Euclidean R-module (dubbed "infinitesimally Euclidean"), for which Jacobi's identity of vector fields with respect to Lie brackets obtains naturally without resorting to such a highly combinatorial device as P. Hall's 42-letter identity. In Section 1.4 we prove Cartan's three magic formulas of Lie derivatives for differential forms synthetically. The remaining two subsections deal with a group G and its action on a microlinear space M, respectively. But that the group G is assumed not only to be microlinear, but also to be infinitesimally Euclidean; we could not even express the fundamental relationship between Ad and ad ("the differential of Ad is ad"). Section 2 is divided into three subsections. The first subsection deals with Hamiltonian vector fields on a symplectic space (M, ω), which are shown to preserve Hamiltonians (i.e., the conservation of energy), to be canonical as infinitesimal transformations, and to constitute a subalgebra of the Lie algebra $\mathscr{X}(M)$ of vector fields on M. The second subsection deals with Poisson brackets. It is shown that the totality $\mathbb{R}^M$ of functions on M is a Lie algebra with respect to Poisson brackets and that the assignment of its Hamiltonian vector field X_H to each function H: $M \to \mathbb{R}$ is a homomorphism of Lie algebras. The third subsection deals with momentum mappings under the action of a group. They are shown to be an invariant under the action and to be infinitesimally equivariant. A fundamental construction of momentum mappings is also given.

As is often the case in expositions of synthetic differential geometry, the reader should presume that we are working in a topos, so that the excluded middle and Zorn's lemma should be avoided. Moerdijk and Reyes (1991) construct toposes eligible for synthetic differential geometry (dubbed "smooth toposes"). Objects of the topos go under such aliases as a "space," a "set," etc.

We are keenly aware that such important topics as Hamilton–Jacobi equations and Poisson manifolds are not even touched. They will be discussed elsewhere.

1. SYNTHETIC DIFFERENTIAL GEOMETRY

Three textbooks on synthetic differential geometry are Kock (1981), Lavendhomme (1987), and Moerdijk and Reyes (1991). Lavendhomme (1987) is most suitable for this paper. If there is anything more than a hasty review of that book in this section, it is our consistent emphasis on the usefulness of the Lie derivative in the context of synthetic differential geometry. This will be particularly so in our treatment of Jacobi's identity of vector fields.

1.1. Real Numbers

Whatever the set $\mathbb{R}$ of real numbers may be, it should be a commutative unitary ring at least. For $\mathbb{R}$ to have plenty of nilpotent infinitesimals coherently, synthetic geometers usually assume the following *generalized Kock–Lawvere axiom*:

(1.1) For any Weil algebra W, the canonical $\mathbb{R}$-algebra homomorphism $W \to \mathbb{R}^{\mathrm{Spec}_{\mathbb{R}}(W)}$ is an isomorphism.

For the definition of a Weil algebra, the reader is referred to Lavendhomme (1987, Chapter II, §1). For some characterizations of a Weil algebra, see Moerdijk and Reyes (1991, Chapter I, 3.17). By way of example,

$$W(n) = \mathbb{R}[x_1, \ldots, x_n]/(x_i x_j)_{1 \leq i \leq n, 1 \leq j \leq n}$$

is a Weil algebra, and $\mathrm{Spec}_{\mathbb{R}}(W(n))$ has the special notation $D(n)$. In particular,

$D(1)$ is usually denoted by D in the literature. Spaces of the form $\mathrm{Spec}_{\mathbf{R}}(W)$ for some Weil algebra W are called *infinitesimal spaces*.

The generalized Kock–Lawvere axiom surely covers the following *Kock–Lawvere axiom* as its origin.

(1.2) The spaces $\mathbf{R}^D$ and $\mathbf{R}^2$ are isomorphic under the assignment $(a, b) \in \mathbf{R}^2 \mapsto \{(d, a + db) \mid d \in D\} \in \mathbf{R}^D$.

The Kock–Lawvere axiom (1.2) is the fulcrum of synthetic differential calculus. Given any $F \in \mathbf{R}^{\mathbf{R}}$ and any $x \in \mathbf{R}$, the function $d \in D \mapsto F(x + d)$ should be of the form $d \in D \mapsto a + db \in \mathbf{R}$ for unique $a, b \in \mathbf{R}$ in virtue of (1.2), in which b is called the *derivative* of F at x and, following the standard notation, is usually denoted by $F'(x)$, $\mathbf{D}F(x)$, etc. The Kock–Lawvere axiom enables us to develop differential calculus without limits and hopefully without tears, for which the reader is referred to Lavendhomme (1987, Chapter I). The following easy consequence of the definition is well known:

(1.3) $(FG)' = F'G + FG'$ for any $F, G \in \mathbf{R}^{\mathbf{R}}$.

This is the prototype of Jacobi's identity for Lie brackets of vector fields and Poisson brackets of functions, the former of which will be discussed in this section and the latter of which will be discussed in the next section. Since synthetic differential calculus is developed solely on the Kock–Lawvere axiom, it is natural that synthetic geometers should have introduced the notion of a *Euclidean space*, which is by definition an R-module subject to the following condition:

(1.4) The spaces M^D and M^2 are isomorphic under the assignment $(x, y) \in M^2 \mapsto \{(d, x + dy) \mid d \in D\} \in M^D$.

1.2. Microlinear Spaces

By appropriately generalizing the generalized Kock–Lawvere axiom (1.1), synthetic geometers have introduced the notion of a *microlinear space*, which is by definition a space M abiding by the following condition:

(1.5) For any good finite limit diagram of Weil algebras $\{W \xrightarrow{\varphi_\lambda} W_\lambda\}$, the induced diagram

$$\{M^{\mathrm{Spec}_{\mathbf{R}}(W)} \xrightarrow{M^{\mathrm{Spec}_{\mathbf{R}}(\varphi_\lambda)}} M_\lambda^{\mathrm{Spec}_{\mathbf{R}}(W_\lambda)}\}$$

is a limit diagram.

Recall that a finite limit diagram of R-algebras is said to be a *good finite limit diagram of Weil algebras* if every object occurring in the diagram is a Weil algebra and every morphism occurring in the diagram is a homomorphism of Weil algebras (i.e., preserving maximal ideals), for which the reader is referred to Lavendhomme (1987, p. 55, "une bonne limite à gauche finie d'algèbres de Weil"). What makes microlinear spaces a curiosity to the community of synthetic differential geometers at large is partly that the class of microlinear spaces is closed under finite limits and that the exponential of a microlinear space by any space is again microlinear, for which the reader is referred to Lavendhomme (1987, Chapter II, §3, Proposition 1). Since R is apparently microlinear, the above closedness of microlinear spaces under finite limits and exponentiation gives rise to microlinear spaces in abundance. Throughout the rest of this section a microlinear space M is arbitrarily chosen and shall be fixed. We denote by Hom(M) and Iso(M) the set of functions from M to M and that of bijections from M onto M, respectively.

1.3. Vector Fields

In synthetic differential geometry a *vector field* on M can be seen from three distinct viewpoints. The most orthodox viewpoint is to regard it as a function $X: M \to M^D$ assigning $X_x \in M^D$ to each $x \in M$ with $X_x(0) = x$ for any $x \in M$. Another viewpoint, committing itself to the classical notion of an infinitesimal flow, is to reckon it as a function $X: M \times D \to M$ with $X(x, 0) = x$ for any $x \in M$. The most radical viewpoint is to look upon it as a function assigning, to each $d \in D$, an *infinitesimal transformation* X_d: $M \to M$ such that X_0 is the identity transformation of M. Throughout the remainder of this paper, unless stated otherwise, we will take the third viewpoint.

We denote by $\mathscr{X}(M)$ the totality of vector fields on M. It is well known to be an R-module in the following sense:

(1.6) Given $X \in \mathscr{X}(M)$ and $a \in$ R, aX is the vector field on M such that $(aX)_d = X_{ad}$ for any $d \in D$.

(1.7) Given $X, Y \in \mathscr{X}(M)$, there exists a unique function $\zeta: D(2) \to M^M$ coinciding with X and Y on the axes. We define $X + Y$ to be the vector field on M such that $(X + Y)_d = \zeta(d, d)$ for any $d \in D$.

In the remainder of this subsection the space M is assumed to be *infinitesimally Euclidean* in the sense that the R-module $\mathscr{X}(M)$ is Euclidean. Given $X \in \mathscr{X}(M)$ and $\varphi \in$ Iso(M), we denote by $\varphi_* X$ the vector field on M assigning, to each $d \in D$, $\varphi \circ X_d \circ \varphi^{-1} \in$ Iso(M). Given $X, Y \in \mathscr{X}(M)$, we denote by $[X, Y]$ the unique vector field on M such that for any $d_1, d_2 \in D$, $[X, Y]_{d_1 d_2}$ is $Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$. It will be shown below, by introducing and exploiting the notion of the Lie derivative $\mathbf{L}_X Y$ of a vector field Y on M with respect to another vector field X on M, that it is a Lie algebra over R, though the maxim itself is already well established in the literature of synthetic differential geometry without assuming that M is infinitesimally Euclidean, for which the reader is referred to Lavendhomme (1987, Chapter III, §2, Proposition 7).

The following useful proposition is quoted from Lavendhomme (1987, Chapter III, §2, Proposition 6).

Proposition 1.1. Let $X, Y \in \mathscr{X}(M)$. The mapping $\lambda: D(2) \to \mathrm{Iso}(M)$ that coincides with X and Y on the axes is given by

$$\lambda(d, d') = X_d \circ Y_{d'} = Y_{d'} \circ X_d$$

Given $X, Y \in \mathscr{X}(M)$, we denote by $\mathbf{L}_X Y$ the unique vector field on M such that

$$(X_{-d})_* Y - Y = d\mathbf{L}_X Y \text{ for any } d \in D$$

The relation between $\mathbf{L}_X Y$ and $[X, Y]$ is simple, as follows.

Theorem 1.2. For any $X, Y \in \mathscr{X}(M)$, $\mathbf{L}_X Y = [X, Y]$.

Proof. For any $d, d' \in D$, we have

$$\begin{aligned}
&((X_{-d})_* Y - Y)_{d'} \\
&= X_{-d} \circ Y_{d'} \circ X_d - Y_{d'} \\
&= Y_{-d'} \circ X_{-d} \circ Y_{d'} \circ X_d \qquad \text{(Proposition 1.1)} \\
&= [X, Y]_{dd'} \\
&= (d[X, Y])_{d'}
\end{aligned}$$

Therefore $(X_{-d})_* Y - Y = d[X, Y]$ for any $d \in D$, which is tantamount to saying that $\mathbf{L}_X Y = [X, Y]$. ■

With the above theorem in mind, we can say that the following theorem is none other than Jacobi's identity of vector fields in disguise.

Theorem 1.3. For all $X, Y, Z \in \mathscr{X}(M)$, we have

(1.8) $\mathbf{L}_X[Y, Z] = [\mathbf{L}_X Y, Z] + [Y, \mathbf{L}_X Z]$.

Proof. It is easy to see that for any $d \in D$,

$$(X_{-d})_*[Y, Z] = [(X_{-d})_* Y, (X_{-d})_* Z]$$

Thus

$$\begin{aligned}
&(X_{-d})_*[Y, Z] - [Y, Z] \\
&= [(X_{-d})_* Y, (X_{-d})_* Z] - [Y, Z] \\
&= [d\mathbf{L}_X Y + Y, d\mathbf{L}_X Z + Z] - [Y, Z] \\
&= d([\mathbf{L}_X Y, Z] + [Y, \mathbf{L}_X Z])
\end{aligned}$$

Therefore $\mathbf{L}_X[Y, Z] = [\mathbf{L}_X Y, Z] + [Y, \mathbf{L}_X Z]$, as was expected. ■

The reader should note that the proof of the above theorem is no more difficult than that of (1.3). Now Jacobi's identity of vector fields with respect to $[\cdot, \cdot]$ is an easy consequence of the above theorems, as we will see just below.

Theorem 1.4. The R-module $\mathscr{X}(M)$ is a Lie algebra with respect to $[\cdot, \cdot]$.

Proof. Aside from trivialities, it suffices to establish the following version of the Jacobi identity:

(1.9) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ for all $X, Y, Z \in \mathscr{X}(M)$.

Since $[X, [Y, Z]] = \mathbf{L}_X[Y, Z]$, $[X, Y] = \mathbf{L}_X Y$, and $[X, Z] = \mathbf{L}_X Z$ by reason of Theorem 1.2, (1.9) is no other than a reformulation of (1.8) of Theorem 1.3. ■

1.4. Differential Forms

Let n and k be natural numbers with $1 \leq k \leq n$. For any $\tau \in M^{D^n}$ and any $a \in \mathbf{R}$, $a_k\tau$ denotes the element of M^{D^n} such that

$$(a_k\tau)(d_1, \ldots, d_n) = \tau(d_1, \ldots, ad_k, \ldots, d_n)$$

for any $(d_1, \ldots, d_n) \in D^n$.

The totality of permutations of numbers $1, 2, \ldots, n$ is denoted by Perm(n). For any $\sigma \in$ Perm(n) we denote by ϵ_σ the parity of σ, which is $+1$ or -1, depending on whether σ is even or odd. Given $\tau \in M^{D^n}$ and $\sigma \in$ Perm(n), we denote by τ^σ the element of M^{D^n} such that

$$\tau^\sigma(d_1, \ldots, d_n) = \tau(d_{\sigma(1)}, \ldots, d_{\sigma(n)})$$

for any $(d_1, \ldots, d_n) \in D^n$.

An *n-form* on M is a function ω: $M^{D^n} \to \mathbf{R}$ satisfying the following conditions:

(1.10) $\omega(a_k\tau) = a\omega(\tau)$ for any $a \in \mathbf{R}$, any $\tau \in M^{D^n}$, and any k $(1 \leq k \leq n)$.

(1.11) $\omega(\tau^\sigma) = \epsilon_\sigma\omega(\tau)$ for any $\sigma \in$ Perm(n) and any $\tau \in M^{D^n}$.

We denote by $\Lambda^n(M)$ the totality of n-forms on M. In particular, $\Lambda^0(M) = \mathbf{R}^M$. It is well known that $\Lambda^n(M)$ is microlinear and Euclidean (Lavendhomme, 1987, Chapter IV, §1, Proposition 2). The union of $\Lambda^n(M)$'s for all $n \in N$ is denoted by $\Lambda(M)$.

Given $\varphi \in$ Hom(M) and $\omega \in \Lambda^n(M)$, we denote by $\varphi^*\omega$ the n-form on M such that

$$(\varphi^*\omega)(\tau) = \omega(\varphi \circ \tau)$$

for any $\tau \in M^{D^n}$. Given $X \in \mathscr{X}(M)$ and $\omega \in \Lambda^n(M)$, we denote by $\mathbf{L}_X\omega$ the n-form on M such that

$$(X_d)^*\omega - \omega = d\mathbf{L}_X\omega$$

for any $d \in D$.

Given $X \in \mathscr{X}(M)$ and $\tau \in M^{D^n}$, we denote by $X * \tau$ the element of $M^{D^{n+1}}$ such that

$$(X * \tau)(d_1, \ldots, d_{n+1}) = X_{d_1}(\tau(d_2, \ldots, d_{n+1}))$$

for any $(d_1, \ldots, d_{n+1}) \in D^{n+1}$. Given $X \in \mathscr{X}(M)$ and $\omega \in \Lambda^{n+1}(M)$, we denote by $\mathbf{i}_x\omega$ the n-form on M such that

$$(\mathbf{i}_X\omega)(\tau) = \omega(X * \tau)$$

for any $\tau \in M^{D^n}$.

We assume knowledge of the synthetic definition of exterior differentiation $\mathbf{d}$ (Lavendhomme, 1987, Chapter IV, §2). What we need about $\mathbf{d}$ in this paper is merely that $\mathbf{d}^2 = 0$ and the following.

Proposition 1.5. Let $\omega \in \Lambda^n(M)$ and $\tau \in M^{D^{n+1}}$. Then

$$(\mathbf{d}\omega)(\tau) = \sum_{i=1}^{n+1} (-1)^{i+1}\mathbf{D}F^i(0)$$

where $F^i(e) = \omega(\tau^i(e))$, with

$$\tau^i(e)(d_1, \ldots, d_n) = \tau(d_1, \ldots, d_{i-1}, e, d_i, \ldots, d_n)$$

Proof. See Lavendhomme (1987, Chapter IV, §2, Proposition 4). ■

Now we establish Cartan's three magic formulas relating **d**, **i**, **L**, and $[\cdot, \cdot]$. The first goes as follows:

Theorem 1.6. For any $X, Y \in \mathscr{X}(M)$, we have

$$\mathbf{L}_{[X,Y]} = \mathbf{L}_X\mathbf{L}_Y - \mathbf{L}_Y\mathbf{L}_X$$

Proof. Let $\omega \in \Lambda(M)$. For any $d_1, d_2 \in D$, we have

$$([X, Y]_{d_1d_2})^*\omega - \omega = d_1d_2\mathbf{L}_{[X,Y]}\omega$$

on the one hand. On the other hand, we have

$$\begin{aligned}&([X, Y]_{d_1d_2})^*\omega - \omega \\ &\quad= (Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1})^*\omega - \omega\end{aligned}$$

$$= (X_{d_1})^*(Y_{d_2})^*(X_{-d_1})^*(Y_{-d_2})^*\omega - \omega$$
$$= (X_{d_1})^*(Y_{d_2})^*(X_{-d_1})^*((Y_{-d_2})^*\omega - \omega)$$
$$+ (X_{d_1})^*(Y_{d_2})^*((X_{-d_1})^*\omega - \omega) + (X_{d_1})^*((Y_{d_2})^*\omega - \omega)$$
$$+ (X_{d_1})^*\omega - \omega$$
$$= -d_2(X_{d_1})^*(Y_{d_2})^*(X_{-d_1})^*\mathbf{L}_Y\omega - d_1(X_{d_1})^*(Y_{d_2})^*\mathbf{L}_X\omega$$
$$+ d_2(X_{d_1})^*\mathbf{L}_Y\omega + d_1\mathbf{L}_X\omega$$
$$= -d_2(X_{d_1})^*(Y_{d_2})^*((X_{-d_1})^*\mathbf{L}_Y\omega - \mathbf{L}_Y\omega)$$
$$- d_2(X_{d_1})^*((Y_{d_2})^*\mathbf{L}_Y\omega - \mathbf{L}_Y\omega) - d_2((X_{d_1})^*\mathbf{L}_Y\omega - \mathbf{L}_Y\omega) - d_2\mathbf{L}_Y\omega$$
$$- d_1(X_{d_1})^*((Y_{d_2})^*\mathbf{L}_X\omega - L_X\omega) - d_1((X_{d_1})^*\mathbf{L}_X\omega - \mathbf{L}_X\omega) - d_1\mathbf{L}_X\omega$$
$$+ d_2((X_{d_1})^*\mathbf{L}_Y\omega - \mathbf{L}_Y\omega) + d_2\mathbf{L}_Y\omega + d_1\mathbf{L}_X\omega$$
$$= d_1d_2(X_{d_1})^*(Y_{d_2})^*\mathbf{L}_X\mathbf{L}_Y\omega - d_1d_2\mathbf{L}_X\mathbf{L}_Y\omega - d_2\mathbf{L}_Y\omega - d_1d_2(X_{d_1})^*\mathbf{L}_Y\mathbf{L}_X\omega$$
$$- d_1\mathbf{L}_X\omega + d_1d_2\mathbf{L}_X\mathbf{L}_Y\omega + d_2\mathbf{L}_Y\omega + d_1\mathbf{L}_X\omega$$
$$= d_1d_2(X_{d_1})^*(Y_{d_2})^*\mathbf{L}_X\mathbf{L}_Y\omega - d_1d_2(X_{d_1})^*\mathbf{L}_Y\mathbf{L}_X\omega$$
$$= d_1d_2(X_{d_1})^*((Y_{d_2})^*\mathbf{L}_X\mathbf{L}_Y\omega - \mathbf{L}_X\mathbf{L}_Y\omega)$$
$$+ d_1d_2((X_{d_1})^*\mathbf{L}_X\mathbf{L}_Y\omega - \mathbf{L}_X\mathbf{L}_Y\omega) + d_1d_2\mathbf{L}_X\mathbf{L}_Y\omega$$
$$- d_1d_2((X_{d_1})^*\mathbf{L}_Y\mathbf{L}_X\omega - \mathbf{L}_Y\mathbf{L}_X\omega) - d_1d_2\mathbf{L}_Y\mathbf{L}_X\omega$$
$$= d_1d_2(\mathbf{L}_X\mathbf{L}_Y\omega - \mathbf{L}_Y\mathbf{L}_X\omega)$$

Therefore $\mathbf{L}_{[X,Y]}\omega = \mathbf{L}_X\mathbf{L}_Y\omega - \mathbf{L}_Y\mathbf{L}_X\omega$, as expected. ■

The following corollary of the above theorem may be of some interest.

Corollary 1.7. For any $\omega \in \Lambda(M)$, any $X, Y \in \mathscr{X}(M)$, and any $d_1, d_2 \in D$, we have

$$([X, Y]_{d_1d_2})^*\omega - \omega = (X_{d_1})^*(Y_{d_2})^*\omega - (Y_{d_2})^*(X_{d_1})^*\omega$$

Proof. This follows from the following calculation

$$([X, Y]_{d_1d_2})^*\omega - \omega$$
$$= d_1d_2\mathbf{L}_{[X,Y]}\omega$$
$$= d_1d_2\mathbf{L}_X\mathbf{L}_Y\omega - d_1d_2\mathbf{L}_Y\mathbf{L}_X\omega \qquad \text{(Theorem 1.6)}$$

$$= d_1\mathbf{L}_X((Y_{d_2})^*\omega - \omega) - d_2\mathbf{L}_Y((X_{d_1})^*\omega - \omega)$$
$$= (X_{d_1})^*((Y_{d_2})^*\omega - \omega) - ((Y_{d_2})^*\omega - \omega)$$
$$- (Y_{d_2})^*((X_{d_1})^*\omega - \omega) + ((X_{d_1})^*\omega - \omega)$$
$$= (X_{d_1})^*(Y_{d_2})^*\omega - (Y_{d_2})^*(X_{d_1})^*\omega \quad \blacksquare$$

The second of Cartan's three magic formulas goes as follows:

Theorem 1.8. For any $X, Y \in \mathscr{X}(M)$, we have

$$\mathbf{i}_{[X,Y]} = \mathbf{L}_X\mathbf{i}_Y - \mathbf{i}_Y\mathbf{L}_X$$

Proof. For any $\omega \in \Lambda(M)$ and any $d \in D$, we have

$$d\mathbf{i}_{[X,Y]}\omega$$
$$= \mathbf{i}_{d[X,Y]}\omega$$
$$= (X_d)^*\mathbf{i}_Y(X_{-d})^*\omega - \mathbf{i}_Y\omega \qquad \text{(Proposition 1.1)}$$
$$= (X_d)^*\mathbf{i}_Y((X_{-d})^*\omega - \omega) + (X_d)^*\mathbf{i}_Y\omega - \mathbf{i}_Y\omega$$
$$= -d(X_d)^*\mathbf{i}_Y\mathbf{L}_X\omega + d\mathbf{L}_X\mathbf{i}_Y\omega$$
$$= -d((X_d)^*\mathbf{i}_Y\mathbf{L}_X\omega - \mathbf{i}_Y\mathbf{L}_X\omega) - d\mathbf{i}_Y\mathbf{L}_X\omega + d\mathbf{L}_X\mathbf{i}_Y\omega$$
$$= d(\mathbf{L}_X\mathbf{i}_Y\omega - \mathbf{i}_Y\mathbf{L}_X\omega)$$

Since d is an arbitrary element of D, we can conclude that $\mathbf{i}_{[X,Y]} = \mathbf{L}_X\mathbf{i}_Y - \mathbf{i}_Y\mathbf{L}_X$, as expected. ■

The last of Cartan's three magic formulas goes as follows:

Theorem 1.9. For any $X \in \mathscr{X}(M)$, we have

$$\mathbf{L}_X = \mathbf{d}\mathbf{i}_X + \mathbf{i}_X\mathbf{d}$$

Proof. Let $\omega \in \Lambda^n(M)$ and $\tau \in M^{D^n}$. On the one hand, we have

$$(\mathbf{L}_X\omega)(\tau) = \mathbf{D}G(0)$$

where $G(e) = \omega(X_e \circ \tau)$ for any $e \in D$. On the other hand, we have

$$(\mathbf{d}\mathbf{i}_X\omega)(\tau) = \sum_{i=1}^{n} (-1)^{i+1}\mathbf{D}F^i(0)$$

$$(\mathbf{i}_X\mathbf{d}\omega)(\tau) = \mathbf{D}G(0) + \sum_{i=1}^{n} (-1)^i\mathbf{D}F^i(0)$$

where $F^i(e) = \omega(X * \tau^i(e))$ for any $e \in D$ with

$$\tau^i(e)(d_1, \ldots, d_{n-1}) = \tau(d_1, \ldots, d_{i-1}, e, d_i, \ldots, d_{n-1})$$

for any $d_1, \ldots, d_{n-1} \in D$. Therefore $(\mathbf{L}_X\omega)(\tau) = (\mathbf{d}\mathbf{i}_X\omega + \mathbf{i}_X\mathbf{d}\omega)(\tau)$, as expected. ■

The proof of the following corollary of the above theorem is standard.

Corollary 1.10. For any $X \in \mathscr{X}(M)$, $\mathbf{d}\mathbf{L}_X = \mathbf{L}_X\mathbf{d}$.

Proof. We have

$$\begin{aligned} \mathbf{d}\mathbf{L}_X &= \mathbf{d}(\mathbf{d}\mathbf{i}_X + \mathbf{i}_X\mathbf{d}) \qquad \text{(Theorem 1.9)} \\ &= \mathbf{d}\mathbf{i}_X\mathbf{d} \\ &= (\mathbf{d}\mathbf{i}_X + \mathbf{i}_X\mathbf{d})\mathbf{d} \\ &= \mathbf{L}_X\mathbf{d} \qquad \text{(Theorem 1.9)} \end{aligned}$$ ■

1.5. The Lie Algebra of a Group

Let G be a group which is microlinear and infinitesimally Euclidean. It shall be fixed throughout the remainder of this section. That G is a group means that it is endowed with the product $(g, h) \in G \times G \mapsto gh \in G$, the unit element 1_G, and the inverse $g \in G \mapsto g^{-1}$. Given $g \in G$, we denote by $\mathscr{L}_g, \mathscr{R}_g$, and $\mathscr{I}_g$ the functions $h \in G \mapsto gh \in G$, $h \in G \mapsto hg \in G$, and $h \in G \mapsto ghg^{-1} \in G$, respectively. A vector field X on G is said to be *left invariant* if $(\mathscr{L}_g)_*X = X$ for any $g \in X$. We denote by $\mathscr{X}_{\mathscr{L}}(G)$ the totality of left-invariant vector fields on G. It is easy to see that $\mathscr{X}_{\mathscr{L}}(G)$ is microlinear and Euclidean. Since $(\mathscr{L}_g)_*[X, Y] = [(\mathscr{L}_g)_*X, (\mathscr{L}_g)_*Y] = [X, Y]$ for any $g \in G$ and any $X, Y \in \mathscr{X}_{\mathscr{L}}(G)$, the R-module $\mathscr{X}_{\mathscr{L}}(G)$ is a subalgebra of the Lie algebra $\mathscr{X}(G)$.

We denote by $\mathfrak{g}$ the totality of tangent vectors of G at 1_G. To put it another way, $\mathfrak{g}$ is the set of all functions $\xi: D \to G$ with $\xi(0) = 1_G$. We can see readily that $\mathfrak{g}$ is an R-module in the following sense:

(1.12) Given $\xi \in \mathfrak{g}$ and $a \in R$, $a\xi$ is the tangent vector of G at 1_G such that $(a\xi)(d) = \xi(ad)$ for any $d \in D$.

(1.13) Given $\xi, \eta \in \mathscr{X}(M)$, there exists a unique function $\zeta: D(2) \to M$ coinciding with ξ and η on the axes. We define $\xi + \eta$ to be the tangent vector of G at 1_G such that $(\xi + \eta)(d) = \zeta(d, d)$ for any $d \in D$.

Tinging our third view of a vector field in Section 1.3 with the first one, we write, given $X \in \mathscr{X}_{\mathscr{L}}(G)$, X_{1_G} for the tangent vector of G at 1_G assigning $X_d(1_G)$ to each $d \in D$. It is easy to see that R-modules $\mathscr{X}_{\mathscr{L}}(G)$ and $\mathfrak{g}$ are isomorphic under the mapping $X \in \mathscr{X}_{\mathscr{L}}(G) \mapsto X_{1_G} \in \mathfrak{g}$, so that $\mathfrak{g}$ is also microlinear and Euclidean. Given ξ, $\eta \in \mathfrak{g}$, we denote by $[\xi, \eta]$ the unique tangent vector of G at 1_G such that for any d_1, $d_2 \in D$,

$$[\xi, \eta](d_1 d_2) = \xi(d_1)\eta(d_2)\xi(-d_1)\eta(-d_2)$$

Now we could mimic the whole discussion of Section 1.3 so as to show that $\mathfrak{g}$ is a Lie algebra with respect to $[\cdot, \cdot]$ just defined. First, as in Proposition 1.1, the microlinearity of G implies readily:

Proposition 1.11. Let $\xi, \eta \in \mathfrak{g}$. The mapping $\lambda: D(2) \to M$ that coincides with ξ and η on the axes is given by

$$\lambda(d, d') = \xi(d)\eta(d') = \eta(d')\xi(d)$$

Given $g \in G$ and $\eta \in \mathfrak{g}$, we denote by $\mathrm{Ad}_g\eta$ the tangent vector of G at 1_G assigning $\mathcal{I}_g\eta(d) \in G$ to each $d \in D$. Just as Proposition 1.1 led to Theorem 1.2, Proposition 1.11 leads to the following theorem.

Theorem 1.12. Given ξ, $\eta \in \mathfrak{g}$, we have

$$\mathrm{Ad}_{\xi(d)}\eta - \eta = \mathbf{d}[\xi, \eta]$$

for any $d \in D$.

We could run this course to the end, but we now conclude this subsection simply by quoting the following theorem from Lavendhomme (1987, Chapter III, §2, Proposition 9).

Theorem 1.13. The isomorphism $X \in \mathscr{X}_{\mathscr{L}}(G) \mapsto X_{1_G} \in \mathfrak{g}$ preserves Lie brackets, i.e., it satisfies the following condition:

(1.14) $[X, Y]_{1_G} = [X_{1_G}, Y_{1_G}]$ for any X, $Y \in \mathscr{X}_{\mathscr{L}}(G)$.

Therefore $\mathfrak{g}$ is a Lie algebra over $\mathbf{R}$ isomorphic to $\mathscr{X}_{\mathscr{L}}(G)$.

The reader should notice that the above theorem was established without assuming that G is infinitesimally Euclidean.

1.6. The Action of a Group

Let G be a microlinear group, which shall be fixed throughout this subsection. An *action* of G on M is a function Φ: $G \times M \to M$ complying with the following conditions:

(1.15) $\Phi(1_G, x) = x$ for any $x \in M$.
(1.16) $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ for any $x \in M$ and any g, $h \in G$.

In the remainder of this section an action Φ shall be fixed. Given $g \in G$, we denote by Φ_g the function from M to M such that $\Phi_g(x) = \Phi(g, x)$

for any $x \in M$. Given $\xi \in \mathfrak{g}$, we denote by ξ_M the vector field on M such that $\xi_{M,d} = \Phi_{\xi(d)}$ for any $d \in D$ and any $x \in M$, where $\xi_{M,d}$ is an alias for $(\xi_M)_d$.

Theorem 1.14. For any $g \in G$ and any $\xi, \eta \in \mathfrak{g}$, we have

(1.17) $(\mathrm{Ad}_g \xi)_M = (\Phi_g)_* \xi_M$.
(1.18) $[\xi, \eta]_M = -[\xi_M, \eta_M]$.

Proof. First we deal with (1.17). For any $d \in D$, we have

$$\begin{aligned}(\mathrm{Ad}_g \xi)_M &= \Phi_{(\mathrm{Ad}_g \xi)(d)} \\ &= \Phi_{g\xi(d)g^{-1}} \\ &= \Phi_g \circ \Phi_{\xi(d)} \circ \Phi_{g^{-1}} \\ &= \Phi_g \circ \Phi_{\xi(d)} \circ \Phi_g^{-1} \\ &= (\Phi_g)_* \xi_M\end{aligned}$$

Now we deal with (1.18). For any $d_1, d_2 \in D$, we have

$$\begin{aligned}[\xi, \eta]_{M,d_1 d_2} &= \Phi_{\xi(d_1)\eta(d_2)\xi(-d_1)\eta(-d_2)} \\ &= \Phi_{\xi(d_1)} \circ \Phi_{\eta(d_2)} \circ \Phi_{\xi(-d_1)} \circ \Phi_{\eta(-d_2)} \\ &= \xi_{M,d_1} \circ \eta_{M,d_2} \circ \xi_{M,-d_1} \circ \eta_{M,-d_2} \\ &= [\eta_M, \xi_M]_{(-d_2)(-d_1)} \\ &= -[\xi_M, \eta_M]_{d_1 d_2}\end{aligned}$$

Therefore $[\xi, \eta]_M = -[\xi_M, \eta_M]$, as expected. ■

2. SYNTHETIC HAMILTONIAN MECHANICS

2.1. Hamiltonian Vector Fields

A *symplectic space* is a pair (M, ω) of a microlinear space M and a 2-form ω on M subject to the following conditions:

(2.1) $\mathbf{d}\omega = 0$.
(2.2) For any 1-form α on M, there exists a unique vector field X on M such that $\mathbf{i}_X \omega = \alpha$.

A symplectic space (M, ω) is arbitrarily chosen and shall be fixed throughout this section. A vector field X on M is called *Hamiltonian* if there exists a function $H: M \to \mathbb{R}$ with $\mathbf{i}_X \omega = \mathbf{d}H$, in which we write X_H for X, H is called a *Hamiltonian function* or an *energy function* for X, and the triple

(M, ω, H) is called a *Hamiltonian mechanical system.* The set of all Hamiltonian vector fields on M is denoted by $\mathscr{X}_{\mathrm{Ham}}(M, \omega)$.

The following is an infinitesimal form of conservation of energy.

Theorem 2.1. Let (M, ω, H) be a Hamiltonian mechanical system. Then $H \circ X_{H,d} = H$ for any $d \in D$, where $X_{H,d}$ is an alias for $(X_H)_d$.

Proof. We have

$$\begin{aligned} H \circ H_{H,d} - H &= (H_{H,d})^*H - H \\ &= d\mathbf{L}_{X_H}H \\ &= d\mathbf{i}_{X_H}\mathbf{d}H \\ &= d\mathbf{i}_{X_H}\mathbf{i}_{X_H}\omega \\ &= 0 \quad \blacksquare \end{aligned}$$

Given a vector field X on M, a function F: $M \to \mathrm{R}$ is called an *integral* for X if $\mathbf{L}_X F = 0$. The above theorem implies that a function H: $M \to \mathrm{R}$ is an integral for the vector field X_H.

A vector field X on M is said to be *locally Hamiltonian* if $\mathbf{L}_X\omega = 0$. This locution is justified by the following proposition.

Proposition 2.2. Any Hamiltonian vector field on M is locally Hamiltonian.

Proof. For any H: $M \to \mathrm{R}$, we have that

$$\begin{aligned} \mathbf{L}_{X_H}\omega &= (\mathbf{i}_{X_H}\mathbf{d} + \mathbf{d}\mathbf{i}_{X_H})\omega \qquad \text{(Theorem 1.9)} \\ &= \mathbf{d}\mathbf{i}_{X_H}\omega \\ &= \mathbf{dd}H \\ &= 0 \quad \blacksquare \end{aligned}$$

We denote by $\mathscr{X}_{\mathrm{LHam}}(M, \omega)$ the totality of locally Hamiltonian vector fields on M. The following is another conservation result in infinitesimal form.

Theorem 2.3. A vector field X on M is locally Hamiltonian iff $(X_d)^*\omega = \omega$ for any $d \in D$.

Proof. Since $(X_d)^*\omega - \omega = d\mathbf{L}_X\omega$, the desired result follows immediately. ■

Given two symplectic microlinear spaces (M_1, ω_1) and (M_2, ω_2), a function φ: $M_1 \to M_2$ is said to be *symplectic* or *canonical* if $\varphi^*\omega_2 = \omega_1$. The above theorem states that transformations X_d ($d \in D$) associated with a locally Hamiltonian vector field X on M are canonical transformations of (M, ω) onto itself. The following is its direct consequence.

Corollary 2.4. If $H: M \to \mathbf{R}$ is a function and X is a locally Hamiltonian vector field on M, then $X_H \circ X_d = X_{(X_d)^*H}$ for any $d \in D$, where X_H and $X_{(X_d)^*H}$ are regarded as functions from M to M^D.

Theorem 2.5. For any $X, Y \in \mathscr{X}_{\mathrm{LHam}}(M, \omega)$, $[X, Y] \in \mathscr{X}_{\mathrm{Ham}}(M, \omega)$.

Proof. Since $\mathbf{L}_X\omega = \mathbf{L}_Y\omega = 0$, we have

$$\begin{aligned}
\mathbf{i}_{[X,Y]}\omega &= \mathbf{L}_X\mathbf{i}_Y\omega - \mathbf{i}_Y\mathbf{L}_X\omega && \text{(Theorem 1.8)}\\
&= \mathbf{L}_X\mathbf{i}_Y\omega\\
&= (\mathbf{d}\mathbf{i}_X + \mathbf{i}_X\mathbf{d})\mathbf{i}_Y\omega && \text{(Theorem 1.9)}\\
&= \mathbf{d}\mathbf{i}_X\mathbf{i}_Y\omega + \mathbf{i}_X\mathbf{d}\mathbf{i}_Y\omega\\
&= \mathbf{d}\mathbf{i}_X\mathbf{i}_Y\omega + \mathbf{i}_X(\mathbf{L}_Y - \mathbf{i}_Y\mathbf{d})\omega && \text{(Theorem 1.9)}\\
&= \mathbf{d}\mathbf{i}_X\mathbf{i}_Y\omega
\end{aligned}$$

Therefore the function $\mathbf{i}_X\mathbf{i}_Y\omega$ on M is eligible to be a Hamiltonian function for the vector field $[X, Y]$. ∎

Corollary 2.6. $\mathscr{X}_{\mathrm{LHam}}(M, \omega)$ is a subalgebra of the Lie algebra $\mathscr{X}(M)$, and $\mathscr{X}_{\mathrm{Ham}}(M, \omega)$ is an ideal of the Lie algebra $\mathscr{X}_{\mathrm{LHam}}(M, \omega)$.

2.2. Poisson Brackets

Now we discuss Poisson brackets. Given two functions $F, G: M \to \mathbf{R}$, their *Poisson bracket* $\{F, G\}$ is defined to be $\mathbf{i}_{X_G}\mathbf{i}_{X_F}\omega$.

Proposition 2.7. If X is a locally Hamiltonian vector field on M and $F, G: M \to \mathbf{R}$ are functions, then $(X_d)^*\{F, G\} = \{(X_d)^*F, (X_d)^*G\}$ for any $d \in D$.

Proof. Follows from Theorem 2.3 and Corollary 2.4. ∎

Proposition 2.8. For any functions $F, G: M \to \mathbf{R}$, we have

$$\{F, G\} = -\mathbf{L}_{X_F}G = \mathbf{L}_{X_G}F$$

Proof. We have

$$\begin{aligned}
\{F, G\} &= \mathbf{i}_{X_G}\mathbf{i}_{X_F}\omega\\
&= \mathbf{i}_{X_G}\mathbf{d}F\\
&= \mathbf{L}_{X_G}F && \text{(Theorem 1.9)}
\end{aligned}$$

Therefore $\{F, G\} = \mathbf{L}_{X_G}F$. Since $\mathbf{i}_{X_G}\mathbf{i}_{X_F}\omega = -\mathbf{i}_{X_F}\mathbf{i}_{X_G}\omega$, the other desired equality $\{F, G\} = -\mathbf{L}_{X_F}G$ follows similarly. ■

The above theorem implies directly the following result.

Corollary 2.9. Let F, H be functions from M to R. Then F is an integral for the vector field X_H iff H is an integral for the vector field X_F.

Our proof of the following theorem, which gives Jacobi's identity of the Poisson bracket $\{\cdot, \cdot\}$ as a direct consequence, is literally infinitesimal and synthetic.

Theorem 2.10. For any functions F, G, H: $M \to \mathbf{R}$, we have

$$\mathbf{L}_{X_H}\{F, G\} = \{\mathbf{L}_{X_H}F, G\} + \{F, \mathbf{L}_{X_H}G\}$$

Proof. We have, on the one hand, that

$$(X_{H,d})^*\{F, G\} - \{F, G\} = d\mathbf{L}_{X_H}\{F, G\}$$

We have, on the other hand, that

$$\begin{aligned}
&(X_{H,d})^*\{F, G\} - \{F, G\} \\
&= \{(H_{H,d})^*F, (X_{H,d})^*G\} - \{F, G\} \qquad \text{(Proposition 2.7)} \\
&= \{d\mathbf{L}_{X_H}F + F, d\mathbf{L}_{X_H}G + G\} - \{F, G\} \\
&= d(\{\mathbf{L}_{X_H}F, G\} + \{F, \mathbf{L}_{X_H}G\})
\end{aligned}$$

Therefore the desired equality follows. ■

Theorem 2.11. The R-module $\mathbf{R}^M$ is a Lie algebra over R with respect to the Poisson bracket $\{\cdot, \cdot\}$.

Proof. It is almost obvious that the function $(F, G) \in \mathbf{R}^M \times \mathbf{R}^M \mapsto \{F, G\}$ is skew-symmetric and bilinear. Thus it remains to establish Jacobi's identity:

(2.3) $\{\{F, G\}, H\} = \{\{F, H\}, G\} + \{F, \{G, H\}\}$ for any $F, G, H \in \mathbf{R}^M$.

This follows from the following calculation:

$$\begin{aligned}
&\{\{F, G\}, H\} \\
&= \mathbf{L}_{X_H}\{F, G\} \qquad \text{(Proposition 2.8)} \\
&= \{L_{X_H}F, G\} + \{F, \mathbf{L}_{X_H}G\} \qquad \text{(Theorem 2.10)} \\
&= \{\{F, H\}, G\} + \{F, \{G, H\}\} \qquad \text{(Proposition 2.8)}
\end{aligned}$$ ■

Now we show that the mapping $F \in \mathbf{R}^M \mapsto X_F \in \mathscr{X}(M)$ is an antihomomorphism of Lie algebras, which would be an easy consequence of Theorem

2.11 if we were generally entitled to identify $\mathscr{X}(M)$ with the derivations of $\mathbb{R}^M$ [cf. Marsden and Ratiu (1994, Proposition 5.5.4)].

To this end we need the notion of the Poisson bracket $\{\alpha, \beta\}$ of 1-forms α, β and its fundamental properties. Given a 1-form α on M, we write $\alpha^\#$ for the unique vector field on M such that $\mathbf{i}_{\alpha^\#}\omega = \alpha$. The condition (2.2) guarantees that the mapping $\alpha \in \Lambda^1(M) \mapsto \alpha^\# \in \mathscr{X}(M)$ is bijective with the inverse mapping $X \in \mathscr{X}(M) \mapsto \mathbf{i}_X\omega \in \Lambda^1(M)$. Given two 1-forms α, β on M, their *Poisson bracket* $\{\alpha, \beta\}$ is defined to be the 1-form $-\mathbf{i}_{[\alpha^\#,\beta^\#]}\omega$. The $\mathbb{R}$-module $\Lambda^1(M)$ inherits a Lie algebra structure from that of $\mathscr{X}(M)$, as we see in the following theorem.

Theorem 2.12. The $\mathbb{R}$-module $\Lambda^1(M)$ is a Lie algebra over $\mathbb{R}$ with respect to the Poisson bracket $\{\cdot, \cdot\}$.

Proof. It is almost obvious that the function $(\alpha, \beta) \in \Lambda^1(M) \times \Lambda^1(M) \mapsto \{\alpha, \beta\}$ is skew-symmetric and bilinear. Thus it remains to establish Jacobi's identity:

(2.4) $\{\{\alpha,\beta\},\gamma\} = \{\{\alpha,\gamma\},\beta\} + \{\alpha,\{\beta,\gamma\}\}$ for any $\alpha, \beta, \gamma \in \Lambda^1(M)$.

This follows from the following calculation:

$$
\begin{aligned}
&\{\{\alpha, \beta\}, \gamma\} \\
&= \{-\mathbf{i}_{[\alpha^\#,\beta^\#]}\omega, \gamma\} \\
&= \mathbf{i}_{[[\alpha^\#,\beta^\#],\gamma^\#]}\omega \\
&= \mathbf{i}_{[[\alpha^\#,\gamma^\#],\beta^\#]}\omega + \mathbf{i}_{[\alpha^\#,[\beta^\#,\gamma^\#]]}\omega \qquad \text{(Theorem 1.4)} \\
&= \{\{\alpha, \gamma\}, \beta\} + \{\alpha, \{\beta, \gamma\}\} \quad \blacksquare
\end{aligned}
$$

Theorem 2.13. Let $\alpha, \beta \in \Lambda^1(M)$. Then

$$\{\alpha, \beta\} = -\mathbf{L}_{\alpha^\#}\beta + \mathbf{L}_{\beta^\#}\alpha + \mathbf{d}\mathbf{i}_{\alpha^\#}\mathbf{i}_{\beta^\#}\omega$$

Proof. This follows from the following calculation:

$$
\begin{aligned}
&\{\alpha, \beta\} \\
&= -\mathbf{i}_{[\alpha^\#,\beta^\#]}\omega \\
&= \mathbf{i}_{\beta^\#}\mathbf{L}_{\alpha^\#}\omega - \mathbf{L}_{\alpha^\#}\mathbf{i}_{\beta^\#}\omega \qquad \text{(Theorem 1.8)} \\
&= \mathbf{i}_{\beta^\#}(\mathbf{d}\mathbf{i}_{\alpha^\#} + \mathbf{i}_{\alpha^\#}\mathbf{d})\omega - \mathbf{L}_{\alpha^\#}\beta \qquad \text{(Theorem 1.9)} \\
&= \mathbf{i}_{\beta^\#}\mathbf{d}\mathbf{i}_{\alpha^\#}\omega - \mathbf{L}_{\alpha^\#}\beta \\
&= (\mathbf{L}_{\beta^\#} - \mathbf{d}\mathbf{i}_{\beta^\#})\mathbf{i}_{\alpha^\#}\omega - \mathbf{L}_{\alpha^\#}\beta \qquad \text{(Theorem 1.9)} \\
&= \mathbf{L}_{\beta^\#}\mathbf{i}_{\alpha^\#}\omega - \mathbf{d}\mathbf{i}_{\beta^\#}\mathbf{i}_{\alpha^\#}\omega - \mathbf{L}_{\alpha^\#}\beta \\
&= \mathbf{L}_{\beta^\#}\alpha + \mathbf{d}\mathbf{i}_{\alpha^\#}\mathbf{i}_{\beta^\#}\omega - \mathbf{L}_{\alpha^\#}\beta \quad \blacksquare
\end{aligned}
$$

Theorem 2.14. Let $F, G \in \mathbf{R}^M$. Then $\mathbf{d}\{F, G\} = \{\mathbf{d}F, \mathbf{d}G\}$.

Proof. This follows from the following calculation:

$$\begin{aligned}
&\{\mathbf{d}F, \mathbf{d}G\}\\
&= -\mathbf{L}_{X_F}\mathbf{d}G + \mathbf{L}_{X_G}\mathbf{d}F + \mathbf{d}\mathbf{i}_{X_F}\mathbf{i}_{X_G}\omega \qquad \text{(Theorem 2.13)}\\
&= -\mathbf{d}(\mathbf{L}_{X_F}G - \mathbf{L}_{X_G}F - \mathbf{i}_{X_F}\mathbf{i}_{X_G}\omega) \qquad \text{(Corollary 1.10)}\\
&= -\mathbf{d}(-\{F, G\} + \{F, G\} - \{F, G\})\\
&= \mathbf{d}\{F, G\} \quad \blacksquare
\end{aligned}$$

Now we are ready to establish the following.

Theorem 2.15. For any $F, G \in \mathbf{R}^M$, $X_{\{F,G\}} = -[X_F, X_G]$

Proof. This follows from the following calculation:

$$\begin{aligned}
X_{\{F,G\}} &= (\mathbf{d}\{F, G\})^{\#}\\
&= \{\mathbf{d}F, \mathbf{d}G\}^{\#} \qquad \text{(Theorem 2.14)}\\
&= -[X_F, X_G] \quad \blacksquare
\end{aligned}$$

2.3. Momentum Mappings

Let G be a microlinear, infinitesimally Euclidean group with its Lie algebra $\mathfrak{g}$, which shall be fixed throughout this subsection. We denote by $\mathfrak{g}^*$ the $\mathbf{R}$-module of all homogeneous functions from $\mathfrak{g}$ to $\mathbf{R}$. For any $g \in G$, we write Ad_g^* for the function from $\mathfrak{g}^*$ to itself assigning, to each $\chi \in \mathfrak{g}^*$, the function $\xi \in \mathfrak{g} \mapsto \chi(\mathrm{Ad}_g\xi) \in \mathbf{R}$.

An action Φ of G on M is called a *symplectic action* of G on (M, ω) if Φ_g is a symplectic transformation of (M, ω) for all $g \in G$. Given a symplectic action of G on (M, ω), a mapping J: $M \to \mathfrak{g}^*$ is called a *momentum mapping* for the action provided that the mapping $\hat{J}: \mathfrak{g} \to \mathbf{R}^M$ defined by $\hat{J}(\xi)(x) = J(x)\xi$ is linear and that $X_{\hat{J}(\xi)} = \xi_M$ for every $\xi \in \mathfrak{g}$. The momentum mapping J is said to be Ad*-*equivriaent* if the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\Phi_g} & M\\
{\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J}\\
\mathfrak{g}^* & \xrightarrow[\mathrm{Ad}^*_{g^{-1}}]{} & \mathfrak{g}^*
\end{array}$$

is commutative for every $g \in G$.

The following conservation law is undoubtedly fundamental.

Theorem 2.16. Let Φ be a symplectic action of G on (M, ω) with a momentum mapping J. Assume a function $H: M \to \mathbf{R}$ to be invariant under the action in the sense that

(2.5) $H(\Phi_g(x)) = H(x)$ for any $x \in M$ and any $g \in G$.

Then J is an integral for X_H in the sense that:

(2.6) $L_{X_H}\hat{J}(\xi) = 0$ for any $\xi \in \mathfrak{g}$.

Proof. The condition (2.5) implies that $d\mathbf{L}_{\xi_M}H = (\Phi_{\xi(d)})^*H - H = 0$ for any $d \in D$, from which we conclude that $\mathbf{L}_{\xi_M}H = 0$. Since $\xi_M = X_{\hat{J}(\xi)}$, Corollary 2.9 implies (2.6), as was desired. ■

The following theorem gives a fundamental means to construct momentum mappings.

Theorem 2.17. Let Φ be a symplectic action of G on (M, ω). Assume the form ω to be exact in the sense that $\omega = -\mathbf{d}\theta$ for some 1-form θ on M. Assume also the action to leave θ invariant in the sense that $(\Phi_g)^*\theta = \theta$ for any $g \in G$. Then the function $J: M \to \mathfrak{g}^*$ defined by

(2.7) $J(x)\cdot\xi = (\mathbf{i}_{\xi_M}\theta)(x)$ for any $x \in M$ and any $\xi \in \mathfrak{g}$

is an Ad*-equivariant momentum mapping for the action.

Proof. Since the action Φ leaves θ invariant, $d\mathbf{L}_{\xi_M}\theta = (\phi_{\xi(d)})^*\theta - \theta = 0$ for any $\xi \in \mathfrak{g}$ and any $d \in D$, which implies that $\mathbf{L}_{\xi_M}\theta = 0$. Since $\mathbf{L}_{\xi_M} = \mathbf{i}_{\xi_M}\mathbf{d} + \mathbf{d}\mathbf{i}_{\xi_M}$ by Theorem 1.9,

$$\mathbf{d}\hat{J}(\xi) = \mathbf{d}\mathbf{i}_{\xi_M}\theta = -\mathbf{i}_{\xi_M}d\theta = \mathbf{i}_{\xi_M}\omega$$

which implies that J is a momentum mapping for the action. The Ad*-equivariance of J follows from (1.17) of Theorem 1.14. ■

Theorem 2.18. Let Φ be a symplectic action of G on (M, ω) with a momentum mapping J. If J is Ad*-equivariant, then $\hat{J}([\xi, \eta]) = \{\hat{J}(\xi), \hat{J}(\eta)\}$ for any $\xi, \eta \in \mathfrak{g}$.

Proof. Since J is Ad*-equivariant,

(2.8) $\Phi_g^*\hat{J}(\eta) = \hat{J}(\mathrm{Ad}_{g^{-1}}\eta)$

for any $g \in G$. Therefore

(2.9) $\Phi_{\xi_d}^*\hat{J}(\eta) - \hat{J}(\eta) = \hat{J}(\mathrm{Ad}_{(\xi_d)^{-1}}\eta) - \hat{J}(\eta)$

for any $d \in D$. For the left-hand side of (2.9) we have

$$\begin{aligned}
&\Phi^*_{\hat{J}_d}\hat{J}(\eta) - \hat{J}(\eta) \\
&\quad = (\xi_{M,d})^*\hat{J}(\eta) - \hat{J}(\eta) \\
&\quad = (X_{\hat{J}(\xi),d})^*\hat{J}(\eta) - \hat{J}(\eta) \\
&\quad = d\mathbf{L}_{X_{\hat{J}(\xi)}}\hat{J}(\eta) \\
&\quad = -d\{\hat{J}(\xi), \hat{J}(\eta)\} \qquad \text{(Proposition 2.8)}
\end{aligned}$$

For the right-hand side of (2.9) we have

$$\begin{aligned}
&\hat{J}(\mathrm{Ad}_{(\xi_d)^{-1}}\,\eta) - \hat{J}(\eta) \\
&\quad = \hat{J}(\mathrm{Ad}_{(-\xi)_d}\eta) - \hat{J}(\eta) \\
&\quad = -d\hat{J}([\xi, \eta]) \qquad \text{[(1.18) of Theorem 1.14]}
\end{aligned}$$

Since $d \in D$ is arbitrary, we can conclude that $\hat{J}([\xi, \eta]) = \{\hat{J}(\xi), \hat{J}(\eta)\}$, as was desired. ■

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Whittaker, E. T. (1961). *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* 4th ed., Cambridge University Press, Cambridge.

NOTE ADDED IN PROOF

(1) The notion and the assumption of infinitesimal Euclideaness such as seen in Section 1.3 are redundant by dint of Proposition 4 of §1 and Proposition 1 of §2 of Lavendhomme (1987, Chapitre III).

(2) After finishing this paper, we got acquainted with the following papers, which should be regarded as precursors of ours:

Minguez, M. C. (1988). Some combinatorial calculus on Lie derivative, *Cahiers de Topologie et Géométrie Différentielle Catégoriques*. **29**, 241–247.

Lavendhomme, R. (1994). Algebre de Lie et groupes microlinéaires, *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, **35**, 29–47.